Angular momentum: $\mathbf{L} = \mathbf{r} \times \mathbf{p} \Rightarrow \dot{\mathbf{L}} = \mathbf{r} \times \mathbf{F} \equiv \Gamma$

$dW \equiv \mathbf{F}\left(\mathbf{r}\right)\cdot d\mathbf{s} \Rightarrow W_{1\to 2} = \displaystyle\int_1^2 d\mathbf{s} \cdot \mathbf{F}\left(\mathbf{r}\right) = \int_1^2 d\mathbf{s} \cdot \left(m\frac{d\mathbf{v}}{dt}\right) = m\int_1^2 dt\frac{d}{dt}\frac{1}{2}v^2 = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$

If $\nabla \times \mathbf{F} = 0$ and $\displaystyle\oint d\mathbf{s} \cdot \mathbf{F}(\mathbf{r}) = 0$, then $\mathbf{F}\left(\mathbf{r}\right) = -\nabla U\left(\mathbf{r}\right)$ and $W = -\displaystyle\int_1^2 d\mathbf{s} \cdot \nabla U(\mathbf{r}) = -U_2 + U_1$
Center of mass: $\displaystyle\sum_i m_i \mathbf{r}_i = \sum_i m_i \mathbf{R} \quad \Rightarrow \quad \mathbf{R} = M^{-1}\sum_i m_i \mathbf{r}_i$
$M\ddot{\mathbf{R}} = \sum\limits_i \dot{\mathbf{p}}_i = \sum\limits_i \mathbf{F}_i^{(e)} + \sum\limits_i \sum\limits_{j\neq i} \mathbf{F}_{ji} = \sum\limits_i \mathbf{F}_i^{(e)} + \frac{1}{2}\sum\limits_{ij}\left(\mathbf{F}_{ij} + \mathbf{F}_{ji}\right) = \sum\limits_i \mathbf{F}_i^{(e)} \equiv \mathbf{F}_i$

0.1 Central Force Motion

$\mathbf{F}(\mathbf{r}) = \hat{r}f(r) \quad \Rightarrow \quad \nabla \times \mathbf{F} = 0$ so $\mathbf{F} = -\nabla V(r) = -\hat{r}\frac{dV(r)}{dr}$
Since the force $\mathbf{F}$ is radial, we have $\dot{\mathbf{L}} = \Gamma = \mathbf{r} \times \mathbf{F} = 0 \quad \Rightarrow \quad \mathbf{l} = \mathbf{r} \times \mathbf{p} =$ constant. Hence the motion is planar since $\mathbf{r} \cdot \mathbf{l} = \mathbf{r} \cdot (\mathbf{r} \times \mathbf{p}) = 0$.
For this conservative force, we can write down the total energy as a constant quantity: $E = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + V(r) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + V(r)$.
For x-y planar motion, $l = l_z = m\left(x\dot{y} - y\dot{x}\right) = mr^2\dot{\phi}$ hence $dA = \frac{1}{2}r(rd\phi) = \frac{1}{2}r^2\dot{\phi}dt \quad \Rightarrow \quad \dot{A} = \frac{l}{2m} = const$, Kepler's 2nd Law.
Now, $\dot{\phi} = \frac{l}{mr^2}$ implies we should define $V_{\text{eff}}(r) = V(r) + \frac{l^2}{2mr^2}$ so that the equation $E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r)$ appears one-dimensional. The two-dimensionality of the orbit rests in the term $\frac{l^2}{2mr^2}$, which acts like a repulsive centrifugal barrier at small r.
Instead of using $\vec{x}, \dot{\vec{x}}$ to describe the motion, we can use $E, \vec{l}, t_0$ and ϕ_0 (this is 6 variables, just like $\vec{x}, \dot{\vec{x}}$) as follows: Solving for dt, $t = \pm\left(\frac{1}{2}m\right)^{1/2}\displaystyle\int^r dr\left[E - V_{\text{eff}}(r)\right]^{-1/2} + t_0$. Invert to get $r(t)$ and solve for ϕ using $l = mr^2\dot{\phi}$ to get $\phi = lm^{-1}\displaystyle\int^t dt\left[r(t)\right]^{-2} + \phi_0$. Ignoring time dependance, the geometric orbit $r(\phi)$ is obtained from $\dot{r} = (dr/d\phi)\dot{\phi} = (dr/d\phi)(l/mr^2)$ so that $E = \frac{m}{2}\left[(dr/d\phi)(l/mr^2)\right]^2 + V_{\text{eff}}(r)$ and $\phi = \pm l(2m)^{-1/2}\displaystyle\int^r drr^{-2}[E - V_{\text{eff}}(r)]^{-1/2} + \phi_0$

0.2 Lagrangian Dynamics

$\dot{p}_i = F_i^{(a)} + R_i$, where $F_i^{(a)}$ is the applied force and R_i is the reaction force (force of constraint). Multiply by the ith virtual displacement and sum over all the coordinates, $\sum_i (F_i^{(a)} + R_i - \dot{(p)}_i)\delta x_i = 0$, where $\displaystyle\sum_i R_i = 0$. In constrained motion, the δx_i are not independant. However, the the δq_i are independant by construction. Hence we convert this equation to generalized coordinates ($n = 3N$) and assume k constraints:

$$
\begin{aligned}
\sum_i \dot{p}_i \delta x_i &= \sum_i F_i \delta x_i \\
\sum_{\sigma=1}^{n-k} \left(\sum_i m_i \frac{d\dot{x}_i}{dt} \frac{\partial x_i}{\partial q_\sigma} \right) \delta q_\sigma &= \sum_{\sigma=1}^{n-k} \left(\sum_{i=1}^{n} F_i \frac{\partial x_i}{\partial q_\sigma} \right) \delta q_\sigma \\
\sum_\sigma \left(\sum_i m_i \left[\frac{d}{dt} \left(\dot{x}_i \frac{\partial x_i}{\partial q_\sigma} \right) - \dot{x}_i \frac{d}{dt} \frac{\partial x_i}{\partial q_\sigma} \right] \right) \delta q_\sigma &= \sum_\sigma Q_\sigma \delta q_\sigma \\
\sum_\sigma \left(\sum_i \left[\frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_\sigma} \left(\frac{1}{2} \sum_i m_i \dot{x}_i^2 \right) \right] - m_i \dot{x}_i \frac{\partial}{\partial q_\sigma} \frac{\partial x_i}{\partial t} \right] \right) \delta q_\sigma &= \sum_\sigma Q_\sigma \delta q_\sigma \\
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} &= Q_\sigma
\end{aligned}
\tag{1}
$$

In the above, one can expand out $\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_\sigma}\right)$ and $\frac{\partial}{\partial q_\sigma}\left(\frac{dx_i}{dt}\right)$ separately to prove that they are equal. Also, the differential $dx_i = \sum_\sigma (\partial x_i/\partial q_\sigma) dq_\sigma + (\partial x_i/\partial t)$ can be used to prove the relation $(\partial x_i/\partial q_\sigma) = (\partial \dot{x}_i/\partial \dot{q}_\sigma)$. Now, $Q_i = \sum_i F_i \frac{\partial x_i}{\partial q_\sigma} = -\sum_i \left[\frac{\partial}{\partial x_i} V(\{x\}, t)\right] \frac{\partial x_i}{\partial q_\sigma} = -\frac{\partial}{\partial q_\sigma} V(q_1, \ldots, q_{n-k}, t)$, hence we obtain

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

where $L = T - V$ for V indepedant of $\{\dot{q}\}$.

Calculus of Variations: The problem is to minimize the integral $\displaystyle\int_{x_1}^{x_2} \phi(y, y', x) dx$. Let $y(x)$ be the function which minimizes the integral and $Y(x) = y(x) + \epsilon\eta(x)$, where ϵ is small and $\eta(x)$ is an arbitrary function which vanishes at the endpoints. Then

$$I(\epsilon) = \int_{x_1}^{x_2} \phi\left[Y(x), Y'(x), x\right] dx = \int_{x_1}^{x_2} \phi(y, y', x) dx + \epsilon \int_{x_1}^{x_2} \left[\frac{\partial \phi}{\partial y} \eta(x) + \frac{\partial \phi}{\partial y'} \eta'(x)\right] dx + O(\epsilon^2)$$

where we have $\left[\frac{dI(\epsilon)}{d\epsilon}\right]_{\epsilon=0}$, whence

$$\int_{x_1}^{x_2} \left[\frac{\partial \phi}{\partial y} \eta(x) + \frac{\partial \phi}{\partial y'} \eta'(x)\right] dx = 0$$

and integrating by parts,

$$\int_{x_1}^{x_2} \eta(x) \left(\frac{\partial \phi}{\partial y} - \frac{d}{dx} \frac{\partial \phi}{\partial y'}\right) dx = 0 \quad \Rightarrow \quad \frac{\partial \phi}{\partial y} - \frac{d}{dx} \frac{\partial \phi}{\partial y'}$$

Alternatively we can define the variations

$$\begin{aligned} Y(x) - y(x) &= \epsilon\eta(x) \equiv \delta y(x) \\ Y'(x) - y'(x) &= \epsilon\eta'(x) \equiv \delta y'(x) \\ \phi\left[Y(x), Y'(x), x\right] - \phi\left[y(x), y'(x), x\right] &\equiv \delta\phi \end{aligned} \tag{2}$$

A Talyor series in $\delta\phi$ is

$$\delta\phi = \frac{\partial\phi}{\partial y}\epsilon\eta + \frac{\partial\phi}{\partial y'}\epsilon\eta' = \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial y'}\delta y'$$

The variation in the integral over $\delta\phi$ is zero,

$$I(\epsilon) = \int_{x_1}^{x_2} \left(\frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial y'}\frac{d}{dx}\delta y \right) dx = \int_{x_1}^{x_2} \delta y \left(\frac{\partial\phi}{\partial y} + \frac{d}{dx}\frac{\partial\phi}{\partial y'} \right) dx = 0$$

$$\Rightarrow \quad \frac{\partial\phi}{\partial y} + \frac{d}{dx}\frac{\partial\phi}{\partial y'} = 0.$$